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An Overview of Separation Axioms by Nearly Open Sets in Topology.

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Abstract: The aim of this paper is to exhibit the research on separation axioms in terms of nearly open sets viz p-open, s-open, α-open & β-open sets. It contains the topological property carried by respective $\wp$ -T_k spaces ($\wp$ = p, s, α & β; k = 0,1,2) under the suitable nearly open mappings . This paper also projects $\wp$ -R_0 & $\wp$ -R_1 spaces where $\wp$ = p, s, α & β and related properties at a glance. In general, the $\wp$ -symmetry of a topological space for $\wp$ = p, s, α & β has been included with interesting examples & results.
Key Words : $\wp$ -T_k spaces, $\wp$ -R_0 & $\wp$ -R_1 spaces & $\wp$ -symmetry .

I. Introduction & Preliminaries:

The weak forms of open sets in a topological space as semi-pre open & b-open sets were introduced by D. Andrijevic through the mathematical papers[1,2]. The concepts of generalized closed sets with the introduction of semi-pre opens were studied by Levine [12] and Njasted [14] investigated α-open sets and Mashour et. al. [13] introduced pre-open sets. The class of such sets is named as the class of nearly open sets by Njasted[14].

After the works of Levine on semi-open sets, several mathematician turned their attention to the generalization of various concepts of topology by considering semi-open sets instead of open sets. When open sets are replaced by semi-open sets, new results were obtained. Consequently, many separation axioms have been formed and studied.

The study of topological invariants is the prime objective of the topology. Keeping this in mind several authors invented new separation axioms. The presented paper is the overview of the common facts of this trend at a glance for researchers.

Throughout this paper, spaces (X, T) and (Y,σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. The notions mentioned in [1,6],[2],[12],[14]&[13] were conceptualized using the closure operator (cl) & the interior operator(int) in the following manner:

* Definition:

A subset A of a topological space (X,T) is called

I. a semi-pre –open[1] or β-open [6] set if A⊆ cl(int(cl(A))) and a semi-pre closed or β-closed if int(cl(int(A))) ⊆ A.
II. a b-open[2] set if A⊆ cl(int(A))∪ int(cl(A)) and a b-closed [8] if cl(int(A)) ∩int(cl(A)) ⊆A.
III. a semi-open [12] set if A⊆ cl(int(A)) and semi-closed if int(cl(A)) ⊆ A.
IV. an α-open[14] set if A⊆ int(cl(int(A))) and an α-closed set if cl(int(cl(A))) ⊆ A.
V. a pre-open [13] set if A ⊆ int(cl (A)) and pre-closed if cl(int(A)) ⊆ A.

The class of pre-open, semi-open ,α –open ,semi-pre open and b-open subsets of a space (X,T) are usually denoted by PO(X,T),SO(X,T), T^{α} , SPO(X,T) & BO(X,T) respectively. Any undefined terminology used in this paper can be known from [4].
In 1996, D.Andrijevic made the fundamental observation:

* Proposition:

For every space (X,T) , PO(X,T) ∪ SO(X,T) ⊆ BO(X,T) ⊆ SPO(X,T) holds but none of these implications can be reversed[10].

* **Proposition : Characterization [10]:**

I. *S is semi-pre-open iff* $S \subseteq sint(sclS)$.
II. *S is semi-open iff* $S \subseteq scl\,(sintS)$.
III. *S is pre-open iff* $S \subseteq pint(pclS)$.
IV. *S is b-open iff* $S \subseteq pcl(pint\,S)$. , *where* $S \subseteq X$ *& (X,T) is a space.*

** The separation axioms of topological spaces are usually denoted with the capital letter "T" after the German word "Trennung" which means separation. Separation axioms are one among the most common important & interesting concepts in Topology and are used to coin more restricted classes of topological spaces. However, the structure and the properties of such spaces are not always that easy to comprehend.

§1. Separation axioms in terms of nearly open sets:

The separation axioms enable us to assert with precision whether a topological space has sufficient number of open sets as well as nearly open sets to serve the purpose that the larger the number of open sets as well as nearly open sets, the greater is the supply of the continuous or respective continuous functions because the concept of continuity or respective continuity is fundamental in analysis & topology and intimately linked with open or nearly open sets.

This section highlights the overview of the separation axioms in terms of nearly open sets viz p-open,s-open ,α- open & β-open sets.

$\wp$ -T_k Topological Spaces ($\wp$ = p, s, α & β; k = 0,1,2):

The literature survey on all $\wp$ -T_k spaces ($\wp$ = p, s, α & β; k = 0,1,2) has been brought under a common frame work.

Definition (1.1): A topological space (X,T) is said to be :

(i) $\wp$ - T_0 space if for each pair of distinct points x and y of X, there exists a $\wp$-open set A such that $x \in A$ but $y \notin B$ & that $y \in A$ but $x \notin B$.

Or

$\wp$- T_0 space if for any two distinct points x and y of X, there exists a $\wp$-open set containing one of them but not the other.

(ii) $\wp$-T_1 space if for each pair of distinct points x and y of X, there exists a pair of $\wp$-open sets A & B such that $x \in A$ but $y \notin A$ & that $y \in B$ but $x \notin B$.

Or

$\wp$- T_1 space if for any pair of distinct points x and y of X, there exist $\wp$ -open sets A & B in the manner that A contains x but not y and B contains y but not x.

(iii) $\wp$- T_2 space if for each pair of distinct points x and y in X, there exist two disjoint $\wp$-open sets A and B such that $x \in A$ & $y \in B$.

Or

$\wp$- T_2 space if for any two distinct points x and y of X, there exist a $\wp$-open sets A& B such that $x \in A$, $y \in B$ and $A \cap B = \varphi$.

Remark (1.1): If a space (X,T) is $\wp$ -T_k , then it is $\wp$ -T_{k-1} , k = 1,2. But the converse is not true.

Example (1.1): Every $\wp$ -T_0 space is not necessarily a $\wp$-T_1 space.
Let us consider the set N of all natural numbers. Let T = { φ, N & G_n ={1,2,3,…..,n},n∈ N}.
Then (N,T) is a topological space. Obviously, every G_n is a $\wp$ -open set where $\wp$ = p, s, α & β.
Clearly, the space (N,T) is a $\wp$ -T_0 space, because if we consider two distinct points m and n (m< n) then G_m = {1,2,3,……m} is a $\wp$ -open set containing m but not containing n and hence it is a $\wp$-T_0 space,.
But it is not a $\wp - T_1$ space because if we choose G_n = {1,2,3,……,n}, then $m \in G_m$ but $n \notin G_m$ and $n \in G_n$ but $m \in G_n$ as m<n.
Hence,(N,T) is not a $\wp$-T_1 space, even though it is a $\wp$-T_0 space.

Example (1.2): Every $\wp$ -T_1 space is not necessarily a $\wp$ -T_2 space.
Let T be the co-finite topology on an infinite set X, then (X,T) is a cofinite topological space.

Obviously, every member of T is a ℘ -open set where ℘ = p, s, α & β**.** Clearly, the space (X,T) is a ℘ **-**T_1 space because if we consider two distinct points x & y of X, then {x}& {y} are finite sets and hence X –{x}, X –{y} are members of T i.e. X-{x}& X-{y} are ℘ **-**open sets such that y ∈ X –{x} & x ∈ X –{y} but x∉ X-{x}& y∉ X-{y}.
But in this case the topological space(X,T) is not ℘ **-**T_2 space.

If possible, let (X,T) be a ℘-T_2 space so that for distinct points x,y there exist ℘-open sets G & H containing x & y respectively in the manner that G ∩ H = φ. Consequently $(G \cap H)^c = \varphi^c$ i.e. $G^c \cup H^c = X$.
Now, G & H being ℘ -open sets , therefore G^c & H^c are both finite by definition of co-finite topology and hence , there union X is also finite . But this contradicts the hypothesis that X is infinite which arises due to our assumption that (X,T) is a ℘-T_2 space. Hence, (X,T) is not ℘-T_2 space.

Example (1.3):
(i) We consider the topological space (X,T) where X = {a,b,c,d} And
T = {φ,{a},{a,b},{c,d},{a,c,d}, X}
Here closed sets are : φ,{b},{a,b},{c,d},{b,c,d}, X.
Simple computations show that
PO(X,T) = {Φ,{a},{c},{d},{a,b},{a,c},{a,d},{c,d},{a,b,c}{a,b,d},{a,c,d}, X}.
SO(X,T) ={ Φ,{a},{a,b},{c,d},{a,c,d},X} = T.
αO(X,T) = T & βO(X,T) = PO(X,T).
Thus (X,T) is p-T_0 & β-T_0 space but neither p-T_k nor β- T_k space where k = 1,2.
Also ,(X,T) is not a ℘ -T_k space where ℘ = s & α; k = 0,1,2.

(ii) Let the topological space (X,T) be given by X = {a,b,c,d} And
T = {φ,{b},{c},{b,c}, X}
Simple computations show that
PO(X,T) = {Φ,{b},{c},{b,c},{a,b,c},{b,c,d}, X}.
SO(X,T)={Φ,{b},{c},{a,b},{a,c},{b,d},{b,c},{c,d},{a,b,c}{a,b,d},{b,c,d},{a,c,d},X}
αO(X,T) = PO(X,T) & βO(X,T) = SO(X,T).
Here (X,T) is s-T_0, s- T_1, s-T_2 space as well as β-T_0,β-T_1 & β-T_2 space. But (X,T) is neither
p-T_k nor α-T_k space where k = 0,1,2.

(iii) Let the topological space (X,T) be illustrated as:
X = {a,b,c,d} and T = {φ,{a},{b},{c},{a,b},{b,c},{c,a},{a,b,c}, X}
Simple computation provides that
PO(X,T) = {Φ,{a},{b},{c},{a,b},{b,c},{c,a},{a,b,c},X}.
SO(X,T) =P(X)-{d}.
αO(X,T) = T
& βO(X,T) = P(X)-{d}.
Here (X,T) is s-T_2 space as well as β-T_2 space. But it is not even p-T_2,α-T_2 space or p-T_1, α-T_1 space. Clearly, (X,T) is a ℘-T_0 space where ℘ = p, s, α & β .

Observations:
(a) In the example (i), (X,T) is not a T_0 -space and also in the example (ii), (X,T) is not a T_0 -space. But in the example (iii) (X,T) is a T_0 –space.
(b) Since, every open set is p-open, s-open ,α –open & β-open , hence
(b_1) (X,T) is T_0 –space ⟹ (X,T) is℘- T_0 –space.
(b_2) (X,T) is T_1 –space ⟹ (X,T) is ℘-T_1 –space.
(b_3) (X,T) is T_2 –space ⟹ (X,T) is ℘- T_2 –space, where ℘ = p, s, α & β .
However, the converse of these results may not be true.
(c) These facts establish that the concepts of ℘-T_k spaces are different from the concepts of T_k -spaces where k = 0,1,2 & ℘ = p, s, α & β .

Theorem (1.1):
A *topological space (X,T) is a ℘ -T_0-space iff for each pair of distinct points x &y of X , the ℘ -cl{x}≠ ℘ -cl{y} where ℘ = p, s, α & β.*

Proof:
Necessity: Let (X,T) be a ℘ -T_0-space where ℘ = p, s, α & β and let x,y be any two distinct point of X. Then we have to show that℘ -cl{x} ≠ ℘ -cl{y}. Since the space is ℘-T_0, there exists a℘ -open set G containing one of them, say x, but not containing y. Then X-G is a℘ -closed set which does not contain x but contains y. By definition℘ -cl{y} is the intersection of all ℘-closed set containing {y}. It follows that ℘ −cl{y}⊂ X – G.
Hence x ∉ X-G implies that x∉ ℘-cl{y}. Thus x∈ ℘cl{x} but x∉ ℘ −cl{y}.
It follows that ℘ -cl{x} ≠ ℘ -cl{y}.

Sufficiency:
Let x ≠ y ⇒ ℘ − cl{x} ≠ ℘ − cl{y} where x,y are points of X. Since ℘ -cl{x} ≠ ℘-cl{y}, there exists at least one point z of X which belongs to one of them, say℘ -cl{x} and does not belong to ℘ −cl{y}. We claim that x ∉ ℘-cl{y}. For if Let x∈ ℘-cl{y}.
Then ℘-cl{x} ⊆ ℘-cl{y} and so x∈ ℘-cl{x} ⊂ ℘-cl{y} which is a contradiction. Accordingly x∉ ℘-cl{y} and consequently x ∈X –(℘-cl{y}) which is ℘ -open .
Hence, X –(℘-cl{y}) is a ℘-open set containing x but not y . It follows that (X,T) is a℘-T_0-space where ℘ = p, s, α & β.

Theorem (1.2): *A topological space (X,T) is a ℘-T_1- space if and only if every singleton subset {x} of X is℘ -closed where ℘ = p, s, α & β.*

Proof: The 'if part': let every singleton subset {x} of X be℘ -closed. We have to show that the space is ℘-T_1. Let x,y, be the two distinct point of X. Then X-{x} is a℘ -open set which contains y but does not contain x. Similarly X-{y} is a℘ - open set which contains x but does not contain y . Hence, the space (X,T) is ℘-T_1 where ℘ = p, s, α & β.

The 'only if ' part : Let the space be ℘-T_1 and let x be any point of X. we want to show that {x} is ℘-closed, that is , to show that X-{x} is℘ -open. Let y∈ X-{x}.Then y≠ x. Since X is℘ -T_1, there exists a ℘-open set G_y such that y∈ G_y but x∉ G_y. It follows that y∈G_y⊂X-{x}.
Hence, X-{x} = ∪{G_y : y∈G_y} = A ℘-open set . i.e. {x} is a ℘-closed set where ℘ = p, s, α & β.

Theorem (1.3): *A space (X,T) is ℘-T_2 space iff for each point x∈X , the intersection of all ℘-closed set containing x is the singleton set {x}, where ℘ = p, s, α & β.*

Proof: Necessity:
Suppose that (X,T) is a ℘-T_2 space where ℘ = p, s, α & β. Then there exist a pair of ℘-open sets G &H for each pair of distinct points x,y in X such that x∈G , y∈H and G ∩ H = φ. Now, G ∩ H = φ ⇒G ⊂H^c. Hence , x∈G⊂H^c so that H^c is a ℘-closed set containing x, which does not contain y as y∈H . therefore y cannot be contained in the intersection of all ℘-closed sets which contains x. Since, y ≠ x is arbitrary, it follows that the intersection of all ℘-closed sets containing x is the singleton set {x}. Consequently, ∩ {F: x∈ F∧ F is ℘-closed} = {x}.

Sufficiency:
Suppose that {x} is the intersection of all ℘-closed subsets of (X,T) containing x where x is an arbitrary point of X.
Let y be any other point of X which is different from x. Obviously, by hypothesis y does not belong to the intersection of all ℘-closed subsets containing x. So there must exist a ℘-closed set , say N, containing x such that y ∉N. Now, N being a ℘-closed nbd of x, there must exist a ℘-open set G such that x∈ G ⊂ N.
Thus, G and N^c are ℘-open sets such that x∈ G, y ∈ N^c and G ∩ N^c = φ.Consequently(X,T) is a ℘ - T_2 space where ℘ = p, s, α & β.
Hence, the theorem.

Remark (1.2): The following example is cited in the support of above three theorems.
Let X = {a,b,c,d,e}. T = {φ,{a},{b},{a,b},{c,d},{a,c,d},{b,c,d},(a,b,c,d},{b,c,d,e},X}.
& T^c = { φ,{a},{e},{a,e},{b,e},{a,b,e},{c,d,e},{a,c,d,e},{b,c,d,e},X}.
ThenβO(X,T)={φ,{a},{b},{c},{d},{a,b},{a,c},{a,d},{b,c},{b,d},{b,e},{c,d},{c,e},{d,e}, {a,b,c},{a,b,d},{a,b,e},{a,c,d},{a,c,e},{b,c,d},{b,c,e},{c,d,e},{a,d,e},{b,d,e},{a,b,c,d}, {a,b,c,e},{b,c,d,e},{a,b,d,e},{a,c,d,e}, X}. = P(X)-{{e},{a,e}}.

$\beta C(X,T) = P(X) - \{\{b,c,d\},\{a,b,c,d\}\}$.
Here, (X,T) is a β-T_2 space, consequently, it is also a β-T_1 and β-T_0 space.
Obviously,
(1) β-cl$\{x\} \neq \beta$-cl$\{y\}$; $\forall$ x,y$\in$ X& x$\neq$ y.
(2) {a},{b},{c},{d},{e} are β-closed sets i.e. every singleton set is β-closed.
(3) $\cap \{F: F\in \beta C(X,T)$ such that $x\in F\} = \{x\} \forall x\in X$.

Definition (1.2): ℘-open mappings:
A mapping f(X,T)→(Y,σ) from one space (X,T) to another space (Y,σ) is called ℘ −open mapping (i.e. ℘ = p,s,α,β) iff f sends ℘-open sets (i.e. ℘ = p,s,α,β –open) of (X,T) into ℘ -open sets (i.e. ℘ = p,s,α,β-open) of (Y,σ).

Theorem (1.4):*The property of a space being a ℘ - T_0 space is a topological property where ℘ = p,s,α&β.*
Proof:
Let f(X,T)→(Y,σ) be a one-one onto & ℘ −open mapping from a ℘ - T_0 space (X,T) to any other topological space (Y,σ). It will be established that (Y,σ) is also a ℘ - T_0 space where ℘ = p,s,α,β.
Let y_1 & y_2 be any two distinct points of Y and as f is one-one & onto, there must exist distinct points x_1 & x_2 of X such that $f(x_1) = y_1$ & $f(x_2) = y_2$……………(1)
Since, (X,T) is a ℘ - T_0 space so there exists a T−℘ −open set G in manner that $x_1\in G$ but $x_2\notin G$.
Again, f, being ℘ −open, provides that f(G) is a σ−℘ −open and containing $f(x_1) = y_1$ and not containing $f(x_2) = y_2$.
Thus, there exists a σ−℘ −open set f(G) which contains y_1 and does not contain y_2 and in turn (Y,σ) is a ℘ - T_0 space
Again, as the property of being ℘ − T_0 space is preserved under one-one , onto & ℘ −open mapping, so it is a topological property.
Hence, the theorem.

Theorem (1.5): *The property of a space being a ℘ - T_1 space is a topological property where ℘ = p, s, α & β.*
Proof:
Let (X,T) be a ℘ - T_1 space and (Y,σ) be any other topological space such that f(X,T)→(Y,σ) is one-one onto & ℘ −open mapping from (X,T) to (Y,σ).
It is required to prove that (Y,σ) is also a ℘ - T_1 space where ℘ = p,s,α,β.
Let y_1 & y_2 be any two distinct points of Y and as f is one-one & onto, there must exist distinct points x_1 & x_2 of X such that $f(x_1) = y_1$ & $f(x_2) = y_2$ ……………(1)
Since, (X,T) is a ℘ - T_1 space so there exists a T−℘ −open sets G & H in manner that $x_1\in G$, $x_2\notin G$ & $x_2\in H$, $x_1\notin H$
……………………..(2)
Again, f, being ℘-open, provides that f(G) & f(H) are σ−℘ −open sets such that
$f(x_1) = y_1 \in f(G)$ but $f(x_2) = y_2 \notin f(G)$.
& $f(x_2) = y_2 \in f(H)$ but $f(x_1) = y_1 \notin f(H)$.
Above relations show that (Y,σ) is also a ℘ - T_1 space.
Again, as the property of being ℘ - T_1 space is preserved under one-one , onto & ℘ −open mapping, so it is a topological property.
Hence, the theorem.

Theorem (1.6): *The property of a space being a ℘ - T_2 space is a topological property where ℘ = p,s,α & β.*
Proof: Let (X,T) be a ℘ −T_2 space and (Y,σ) be any other topological space such that f(X,T)→(Y,σ) is one-one onto & ℘ −open mapping from (X,T) to (Y,σ).
It is required to prove that (Y,σ) is also a ℘ - T_2 space where ℘ = p,s,α &β.
Let y_1 & y_2 be any two distinct points of Y and as f is one-one & onto, there must exist distinct points x_1 & x_2 of X such that $f(x_1) = y_1$ & $f(x_2) = y_2$ ……………(1)
Since, (X,T) is a ℘ - T_2 space so there exists a T−℘ −open set G & H such that $x_1\in G$, $x_2\notin G$ & $G \cap H = \varphi$ ………………..(2)
Again, f, being ℘ −open, provides that f(G) & f(H) are σ−℘ −open sets such that

$$f(x_1) = y_1 \in f(G) \text{ but } f(x_2) = y_2 \in f(H) \text{ and}$$
$$\& \; G \cap H = \varphi \Rightarrow f(G \cap H) = \varphi \Rightarrow f(G) \cap f(H) = \varphi.$$

Above relations show that (Y,σ) is also a ℘ - T_2 space.

Again, as the property of being ℘ **-** T_2 space is preserved under one-one , onto & ℘**-**open mapping, so it is a topological property.
Hence, the theorem.

§2. ℘ - R_0 spaces where ℘ = p,s,α & β.

In this section, the notion of ℘ **-** $\mathbf{R_0}$ spaces where ℘ stands for p,s,α,β is introduced and some basic properties are discussed. But before we take up it, we project the notion of the ℘ -kernel of a set A of a space (X,T) and the ℘ -kernel of a point x of a space (X,T) in the following manner:

Definition (2.1):

In a topological space (X,T), *if* $A \subseteq X$, *then the* ℘ − *kernel of* A, *denoted by* ℘ − $ker(A)$, *is defined to be the set* ℘ − $ker(A) = \cap\{\mathcal{O} \in \wp O(X,T) \mid A \subseteq \mathcal{O}\}$.

Definition (2.2):

If x be a point of a topological space (X,T), then the ℘ -kernel of x , denoted by ℘ -ker({x}) is defined to be the set ℘ − ker({x})= $\cap\{ \mathcal{O} \in \wp O(X,T) \mid x \in \mathcal{O} \}$.

Lemma (2.1):

If A be a subset of of a topological space (X,T), then ℘ −ker(A) = ∩{x∈ X |℘ -cl({x})∩ A ≠ Φ}.

Proof: Let x∈ ℘ −ker(A) where A⊆ X & (X,T) is a topological space. On the contrary, we assume that ℘ -cl({x})∩ A = Φ. Hence ,x ∉ X – {℘ -cl({x})} which is a ℘-open set containing A. This is impossible as x ∈ ℘ −ker(A). Consequently, ℘ -cl({x})∩ A≠ Φ.
Again let , ℘ -cl({x})∩ ≠ Φ exist and at the same time let x ∉ ℘ −ker(A). This means that there exists a ℘-open set B containing A and x ∉ B.
Let y ∈ ℘ -cl({x})∩ A. therefore, B is a ℘ −nbhd of y for which x ∉ B. By this contradiction, we have x ∈ ℘-ker(A).
Hence, p-ker({A})=∩ {x∈ X |℘ -cl({x})∩ A ≠ Φ}.

Definition (2.3): ℘ − $\mathbf{R_0}$ spaces:

A topological space (X,T) is said to be a ℘ − R_0 space if every ℘ − open set contains the ℘ − closure of each of its singletons, where ℘ = p,s,α & β.
The implications between ℘ − R_0 spaces are indicated by the following diagram:

R_0 space	⟹	α-R_0 space	⟹	s-R_0 space
		⇓		⇓
		p- R_0 space		β- R_0 space.

We, however, know that a R_0-space is a topological space in which the closure of the singleton of every point of an open set is contained in that set.
None of the above implications in the diagram is reversible, as illustrated by the following examples:

Example (2.1):

Let X = {a,b,c}, T = {φ,(a,b},X}. Then PO(X,T) = {φ,{a},{b},{a,b},{b,c},{c,a},X}.
& PC(X,T) = {φ,{b},{a},{c},{a,c},{b,c},X}.
Hence, (X,T) is a p- R_0 space.
Again, αO(X,T) = {φ, {a,b},X} = sO(X,T).
& αC(X,T) = {φ,{c},X} = sC(X,T).
Since, α-cl({a}) = X ⊄{a,b}∈ αO(X,T), hence,(X,T) is not a α-R_0 space.
Similar is the reason for (X,T) to be not a s-R_0 space.

Example (2.2):

Let X = {a,b,c} , T = {φ,{a},{b},{a,b},X}.
Then sO(X,T) = {φ,{a},{b},{a,b},{b,c},{c,a},X} = βO(X,T).
& sC(X,T) = {φ,{b},{a},{c},{a,c},{b,c},X} = βC(X,T).
Hence, (X,T) is a s-R_0 space as well as β-R_0 space.

Again, PO(X,T) = {φ,{a},{b},{a,b},X} = αO(X,T).
& PC(X,T) ={φ,{c},{a,c},{b,c},X} = αC(X,T).

Since, pcl({a}) = {a,c}⊄{a,b}∈ PO(X,T), hence, (X,T) is not a p-R_0 space.
Similar is the reason for (X,T) to be not an α-R_0 space.

Remark (2.1):
(1) The concepts of p-R_0 space and s-R_0 are independent. Example (2.1) shows that the space (X,T)is p-R_0 but s-R_0 where as in example (2.2), the space (X,T) is s-R_0 but not p-R_0.
(2) The notion of α-R_0 does not imply the notion of R_0 as it is shown by the following example.

Example (2.3):
Let X be an infinite set and p∈X be a fixed point. Let T = {φ &G ⊂ X –{p}& G^c is finite}.
It can be observed that if G is an open set and x∈ G, then cl({x}) = X ⊄G. So, (X,T) is not a R_0 space but as X is α- T_1 so every {x} is α-closed so α-cl({x}) = {x} ⊂G, ∀ x ∈ G & G ∈ αO(X,T). Hence, (X,T) is an α-R_0 space.

Example(2.4):
Let X = {a,b,c,d,e}. T = {φ,{a},{b},{a,b},{c,d},{a,c,d},{b,c,d},(a,b,c,d},{b,c,d,e},X}.
& T^c = { φ,{a},{e},{a,e},{b,e},{a,b,e},{c,d,e},{a,c,d,e},{b,c,d,e},X}. Then
PO(X,T)={φ,{a},{b},{c},{d},{a,b},{a,c},{a,d},{b,c},{b,d},{c,d},{a,b,c},
{a,b,d},{a,c,d},{b,c,d},{b,c,e},{a,b,c,d},{a,b,c,e},{b,c,d,e},{a,b,d,e}X}.
PC(X,T)={φ,{a},{c},{d},{e},{a,d},{a,e},{b,e},{c,e},{d,e},{a,b,e},{a,c,e},{a,d,e},{b,c,e},{b,d,e},
{c,d,e},{a,b,c,e},{a,b,d,e},{a,c,d,e},{b,c,d,e},X}.
Since,p-cl({b})= {b,e} ⊄ {a,b,c,d}∈ PO(X,T), Hence, (X,T) is not a p-R_0 space.
Again, βO(X,T) = P(X)-{{e},{a,e}} & βC(X,T) = P(X) – {{b,c,d},{a,b,c,d}}.
Since, (X,T) is a β-T_1 space so every {x} is β-closed which means that β-cl({x}) = {x} ⊂G∀ x ∈ G & G∈βO(X,T). Consequently, (X,T) is a β-R_0 space.
Next,SO(X,T)={φ,X,{a},{b},{a,b},{b,e},{c,d},{a,b,e},{a,c,d},{b,c,d},{c,d,e},{a,b,c,d},{b,c,d,e},{a,c,d,e}}.
SC(X,T)={φ,X,{a},{b},{e},{a,b},{a,e},{b,e},{c,d},{a,b,e},{a,c,d},{c,d,e},{a,c,d,e},{ b,c,d,e}}.
Here, (X,T) is a s-R_0 space.
Also, αO(X,T) = T & αC(X,T) = { φ,X,{a},{e},{a,e},{b,e},{a,b,e},{c,d,e},{a,c,d,e},{b,c,d,e}}.
Here, (X,T) is not an α-R_0 space.
We, now, mention the following lemmas with proofs, useful in the sequel.

Lemma (2.2): *In a topological space (X,T), for each pair of distinct points*
x,y, ∈ X, x ∈ ℘ −cl({y}) ⇔ y ∈ ℘ −ker({x}), where ℘ = p,s, α & β.

Proof:
Suppose that y∉ ℘ −ker({x}). Then there exists a ℘ -open set V containing x such that y ∈ V. Therefore, we have x∉ ℘ −cl({y}).
This means that $y \notin \wp - ker(\{x\}) \Rightarrow x \notin \wp - cl(\{y\})$.
$i.e.$ $\urcorner x \notin \wp - cl(\{y\}) \Rightarrow \urcorner y \notin \wp - ker(\{x\})$
$i.e.$ $x \in \wp - cl(\{y\}) \Rightarrow y \in \wp - ker(\{x\})$.

Similar is the argument for the proof of the converse i.e.
y ∈ ℘ −ker({x})⇒ x∈ ℘ −cl({y}).
Hence, the theorem.

Lemma (2.3): *The following statement are equivalent for each pair of points x & y in a topological space (X,T):*
(a) $\wp - ker(\{x\}) \neq \wp - ker(\{y\})$.
(b) $\wp - cl(\{x\}) \neq \wp - cl(\{y\}). Where\ \wp = p, s, \alpha \,\&\, \beta$.

Proof: (a) ⇒(b):
$Suppose\ that\ \wp - ker(\{x\}) \neq \wp - ker(\{y\}), then\ there\ exists\ a\ point\ z\ in\ X\ such\ that\ z \in \wp - ker(\{x\})\ and\ z \notin \wp - ker(\{y\})$.
Since $z \in \wp - ker(\{x\}), hence\ x \in \wp - cl(\{z\})$. This means that
$\{x\} \cap \in \wp -$ cl({z}) ≠ φ.
$By\ z \notin \wp - ker(\{y\}), we\ have\ \{y\} \cap \wp - cl(\{z\}) =$ φ.
Since, x ∈ ℘-cl({z}), ℘-cl({x}) ⊂ ℘-cl({z}) and {y}∩ ℘-cl({x}) = φ.
Hence, ℘-cl({x}) ≠ ℘-cl({y}).

(b) ⇒(a): Suppose that ℘-cl({x}) ≠ ℘-cl({y}). Then there exists a point z in X such that z ∈ ℘-cl({x}) and z ∉ ℘ −cl({y}).There, there exists a ℘ -open set containing z and therefore x but not y i.e. y ∉ ℘-ker({x}). Hence, ℘ −ker({x}) ≠ ℘ −ker({y}).
Hence, the theorem.

Theorem (2.1):
A space (X,T) is℘ -R_0 space if and only if for each pair x,y of distinct points in X,
℘-cl({x}) ∩ ℘-cl({y}) = φ or {x,y} ⊂ ℘-cl({x}) ∩ ℘-cl({y}) where ℘ = p,s, α & β.

Proof: Necessity :
Let (X,T) be a℘ -R_0 space and x,y ∈X, x ≠y. On the contrary, suppose that
℘-cl({x}) ∩ ℘-cl({y}) ≠ φ & {x,y} ⊄ ℘-cl({x}) ∩ ℘-cl({y}) .
Let z ∈ ℘-cl({x}) ∩ ℘-cl({y}) & x ∉ ℘-cl({x}) ∩ ℘-cl({y}).
Then x ∉ ℘-cl({y}) and x ∈(p-cl{y})^c which is a ℘ -open set. But ℘-cl({x}) ⊄ [℘-cl({y})]^c.
which appears as a contradiction as (X,T) is a ℘-R_0 space.
Hence, for each pair of distinct points x,y of X, we have ℘-cl({x}) ∩ ℘-cl({y}) = φ or
{x,y}⊂ ℘-cl({x}) ∩ ℘-cl({y}) .

Sufficiency :
Let U be a ℘ -open set and x ∈U. Suppose that ℘-cl({x}) ⊄ U. So there is a point
y∈ ℘-cl({x}) such that y ∉ U and ℘-cl({y}) ∩ U = φ.
Since, U^c is ℘ -closed & y ∈ U^c, hence, {x,y} ⊄ ℘-cl({y}) ∩ ℘-cl({x}) and thus
℘-cl({x}) ∩ ℘-cl({y}) ≠ φ . Consequently, the assumption of the condition provides that (X,T) is ℘- R_0 space.
Hence, the theorem.

Theorem (2.2): *For a topological space (X,T), the following properties are equivalent :*
(a) (X,T) *is a*℘ $-R0$ *space*;
(b) $\wp - cl(\{x\}) = \wp - ker(\{x\}), \forall\ x \in X, where\ \wp = p, s, \alpha\ \&\ \beta.$

Proof: (a) ⇒(b):
Let (X,T) be a℘ $-R0\ space.$
$By\ definition\ (2.2), for\ any\ x \in X, we\ have\ \wp - ker(\{x\}) = \cap\{\, O \in \wp O(X,T) \mid x \in O\}.$

And by definition (2.3) , each ℘ -open set θ containing x contains ℘ −cl({x}).
Hence, $\wp - cl(\{x\}) \subset \wp - ker(\{x\})$.
$Let\ y \in \wp - ker(\{x\}), then\ x \in \wp - ker(\{y\})\ by\ lemma (2.2), and\ so\ \wp - cl(\{x\}) = \wp - cl(\{y\}). Therefore, y \in \wp - cl(\{x\}). These\ mean\ that\ \wp - ker(\{x\}) \subset \wp - cl(\{x\}).$
$Hence, \wp - cl(\{x\}) = \wp - ker(\{x\}).$
(b)⇒(a):
Suppose that for a topological space $(X,T), \wp - cl(\{x\}) = \wp - ker(\{x\})\ \forall\ x \in X.$
Let G be any ℘ −open set in (X,T) , then for every $p \in G, \wp - ker(\{p\}) = \cap\{\, G \in \wp O(X,T) \mid p \in G\}. But\ \wp - cl(\{p\}) = \wp - ker(\{p\})$ by hypothesis. Hence, combing these two, we observe that for every p∈G∈ ℘O(X,T), ℘-cl({x})∈G. Consequently, (X,T) is a ℘- R_0 space.
Hence, the theorem.

Theorem (2.3): *For a topological space (X,T) , the following properties are equivalent:*
(a) (X,T) is a ℘ −R_0 space.
(b) If F is ℘-closed, then F = ℘- ker (F);
(c) If F is ℘-closed and x ∈ F, then ℘- ker ({x}) ⊂ F.
(d) If x∈ X, then ℘-ker({x}) ⊂ ℘-cl({x}).

Proof: (a) ⇒(b):

(a) Let (X,T) is a $\wp-R_0$ space & F, a $\wp$-closed set. Let x ∉ F, then F^c is a $\wp$ -open set containing x, so that $\wp$-cl({x}) ⊂ F^c as (X,T) is a $\wp-R_0$. This means that $\wp$-cl({x})∩ F= φ and by lemma (2.1), x ∉ $\wp$- ker (F).

Therefore, $\wp$- ker (F) = F.

(b) ⇒(c):

In general, A ⊂ B ⇒ $\wp$- ker (A) ⊂ $\wp$- ker (B), Hence, it follows that for x ∈ F, {x} ⊂ F ⇒ $\wp$- ker({x}) ⊂ $\wp$- ker(F) = F as F is $\wp$-closed.

(c) ⇒(d):

Since, x ∈ $\wp$-cl({x}) and $\wp$-cl({x}) is $\wp$ -closed, hence, using (c) we get $\wp$- ker({x}) ⊂ $\wp$- cl({x}).

(d)⇒(a) :

Let (X,T) be a topological space in which $\wp$- ker({x}) ⊂ $\wp$- cl({x}) for every x ∈ X.

Let y ∈ $\wp$-cl({x}), then x ∈ $\wp$-ker({y}), since, y ∈ $\wp$-cl({y}) and $\wp$-cl({y}) is $\wp$-closed, by hypothesis x ∈ $\wp$-ker({y})⊂ $\wp$-cl({y}). Therefore y ∈ $\wp$-cl({x}) ⇒ x ∈ $\wp$-cl({y}). Similarly, x ∈ $\wp$-cl({y}) implies y ∈ $\wp$-cl({x}). Thus (X,T) is $\wp$-R_0 space, using theorem (2.4).

Hence, the theorem.

Theorem (2.4): *for a topological space (X,T) , the following properties are equivalent:*

(a) (X,T) *is a* $\wp - R0$ *space;*

(b) *For any points* x & y *of* $X, x \in \wp - cl(\{y\}) \Leftrightarrow y \in \wp - cl(\{x\})$.

Proof: (a) ⇒(b):

Let (X,T) be a ∈ $\wp-R_0$ space. Let x & y be any two points of X. Assume that x ∈ $\wp$ −cl({y}) and D is any $\wp$ −open set such that y∈ D.

Now, by hypothesis, x ∈ D. Therefore, every $\wp$ −open set containing y contains x. Hence, y ∈ $\wp$ −cl({x}) i.e. $x \in \wp - cl(\{y\} \Rightarrow y \in \wp - cl(\{x\})$. The converse is obvious and $x \in \wp - cl(\{y\}) \Leftrightarrow y \in \wp - cl(\{x\})$.

(b) ⇒(a):

Let U be $\wp$-open set and x ∈ U. If y ∉ U, then x ∉ $\wp$-cl({y}) and hence, y ∉ $\wp$ −cl({x}). This implies that $\wp$ −cl({x}) ⊂ U. Hence, (X,T) is a $\wp$-R_0 space.

Hence, the theorem.

§3. $\wp - R1$ *spaces where* $\wp = p, s, \alpha$ & β.

This section includes the notion of $\wp-R_1$ spaces where $\wp$ stands for p,s, α & β and their basic properties.

Definition (3.1):

A topological space (X,T) is said to be a $\wp$-R_1 space if for each pair of distinct points x & y of X with $\wp$-cl $(\{x\}) \neq$
$\wp - cl(\{y\})$ *, there exist disjoint pair of* $\wp$ *− open sets U and V such that* $\wp - cl(\{x\})\ U$ & $\wp - cl(\{y\}) \subset V$, *where* $\wp = p, s, \alpha$ & β.

Theorem (3.1): *If (X,T) is a* $\wp$*-*R_1 *space, then it is a* $\wp$*-*R_0 *space.*

Proof: Suppose that (X,T) is a $\wp$-R_1 space where $\wp$ = p,s, α & β.

Let U be a $\wp$-open set and x ∈U. then for each point y∈U^c, $\wp$-cl ({x}) ≠ $\wp$-cl({y}).

Since,(X,T) is a $\wp$-R_1 space, there exist a pair of $\wp$-open sets U_y & V_y such that $\wp$-cl ({x}) ⊂ U_y & $\wp$-cl ({y}) ⊂ V_y & $U_y \cap V_y$ = φ.

Let A = ∪{V_y: y ∈U^c}. Then U^c ⊂A, x ∈ A and A is a $\wp$-open set.

Therefore, $\wp$-cl({x}) ⊂A^c ⊂ U which means that (X,T) is a $\wp$-R_0 space.

Hence, the theorem.

Example (3.1):

If p be a fixed point of (X,T) with T as the co-finite topology on X given as
T = {φ, X,G with G ⊂ X – {p} & G^c is finite.}, then the space (X,T) is $\wp$-R_0 but it is not $\wp$-R_1 where $\wp$ = p,s, α & β.

Theorem (3.2):

A space (X,T) is a $\wp -R_1$ space iff for each pair of distinct points x & y of X with $\wp -cl(\{x\}) \neq \wp -cl(\{y\})$, there exist disjoint pair of $\wp$ −open sets U and V such that $x \in U, y \in V$ & $U \cap V = \varphi$.

Necessity:
Let (X,T) be a $\wp -R_1$ space. By definition (3.1), for each pair of distinct points x & y of X with $\wp - cl(\{x\}) \neq \wp$-cl({y}) ,there can always be obtained disjoint pair of $\wp$-open sets U and V such that $\wp - cl(\{x\}) \subset U$ & $\wp - cl(\{y\}) \subset V$ where $U \cap V = \varphi$. We, however, know that $p \in \wp - cl(\{p\}), \forall p \in X$. Hence, $x \in U, y \in V$ & $U \cap V = \varphi$.

Sufficiency:
Let $x, y \in X$ and $x \neq y$ such that $\wp - cl(\{x\}) \neq \wp - cl(\{y\})$. Also let U & V be disjoint $\wp$ -open sets for which $x \in U, y \in V$.
Since, $U \cap V = \varphi$, hence, $x \in \wp - cl(\{x\}) \subset U$ & $y \in \wp - cl(\{y\}) \subset V$.Consequently, (X,T) is a $\wp -R_1$ space. Hence, the theorem.

Corollary (3.1):
Every $\wp -T_2$ space is $\wp -R_1$ space, but the converse is not true. However, we have the following result.

Theorem (3.3): *Every $\wp -T_1$ & $\wp -R_1$ space is $\wp -T_2$ space.*

Proof:
Let (X,T) be a $\wp -T_1$ as well as $\wp -R_1$ space. Since, (X,T) is a $\wp -T_1$ space, hence, $\wp -cl(\{x\}) = \{x\} \neq \{y\} = \wp -cl(\{y\})$ for $x,y \in X$ & $x \neq y$.
Now, theorem (3.2) provides that as (X,T) is a $\wp -R_1$ space and here, $x, y \in X$ and $x \neq y$ such that $\wp -cl(\{x\}) \neq \wp -cl(\{y\})$, so there exist $\wp$ -open sets U & V such that $x \in U, y \in V$ & $U \cap V = \varphi$. Consequently, (X,T) is a $\wp -T_2$ space.
Hence, the theorem.

Theorem (3.4): *For a topological space (X,T) , the following properties are equivalent:*
(a) (X,T) is a $\wp -R_1$ space;
(b) For any two distinct points $x,y \in X$ with $\wp -cl(\{x\}) \neq \wp -cl(\{y\})$, there exist $\wp$ -closed sets F_1 & F_2 such that $x \in F_1$, $y \in F_2$ $x \notin F_2$, $y \notin F_1$ and $F_1 \cup F_2 = X$, where $\wp$ = p,s, α & β.

Proof: (a) ⇒(b):
Suppose that (X,T) is a $\wp -R_1$ space. Let $x,y \in X$ and $x \neq y$ and with $\wp -cl(\{x\}) \neq \wp -cl(\{y\})$,by Theorem (3.2), there exist $\wp$ -open sets U & V such that $x \in U, y \in V$. Then, $F_1 = V^c$ *is* a $\wp$ -closed set & $F_2 = U^c$ is also $\wp$ -closed set such that $x \in F_1$, $y \in F_2$ $x \notin F_2$, $y \notin F_1$ and $F_1 \cup F_2 = X$.
(b)⇒(a):
Let $x,y \in X$ such that $\wp -cl(\{x\}) \neq \wp -cl(\{y\})$.This means that $\wp -cl(\{x\}) \cap \wp -cl(\{y\}) = \varphi$.
By the assume condition (b), there exist $\wp$ -closed sets F_1 & F_2 such that $x \in F_1$, $y \in F_2$, $x \notin F_2$, $y \notin F_1$ and $F_1 \cup F_2 = X$.
Therefore, $x \in F_2^c = U$ = A $\wp$ -open set.
& $y \in F_1^c = V$ = A $\wp$ -open set.
Also $U \cap V = \varphi$.
These facts indicates that
$x \in \wp$-cl ({x}) ⊂ U & $y \in \wp$-cl ({y}) ⊂ V such that $U \cap V = \varphi$. Consequently, (X,T) is a $\wp$-R_1 space.
Hence the theorem.

§4. $\wp$ −symmetry of A space & $\wp$ − generalized closed set:
We, now, define $\wp$ − symmetry of a space & (X,T) & $\wp$ −generalized closed set (briefly $\wp$g-closed set) in a space (X,T) as:

Definition (4.1): A space(X,T) is said to be $\wp$ −symmetric if for every pair of points x,y. in X , $x \in \wp -cl(\{y\}) \Rightarrow y \in \wp -cl(\{x\})$ where $\wp$ = p,s, α & β.

Definition (4.2): A subset A of a space (X,T) is said to be a $\wp$ −generalized closed set(briefly $\wp$g-closed set) if $\wp -cl(\{A\}) \subseteq U$ whenever $A \subseteq U$ & U is $\wp$ −open in X where$\wp$ = p,s, α & β.

Lemma (4.1): *Every $\wp$ -closed set is a $\wp$g-closed set but the converse is not true where $\wp$ = p,s, α & β.*

Proof:
It follows from the fact that whenever A is $\wp$ −closed set, we have$\wp$-cl(A) = A for $\wp$ = p,s, α & β,so the criteria $\wp$ −cl ({A}) ⊆ U whenever A⊆ U & U is $\wp$ − open exists & A turns to be a $\wp$g-closed set.
But the converse need not to be true as illustrated by the following example:
Let X = {a,b,c,d} And T = {φ,{a},{a,b},{c,d},{a,c,d}, X}
Here closed sets are : φ,{b},{a,b},{c,d},{b,c,d}, X.
Then , T_s ={ Φ,{a},{a,b},{c,d},{a,c,d},X}, & T_s^C = {φ,{b},{a,b},{c,d},{b,c,d},X}.

Now, T_{sg}^C = the class of all sg-closed sets.
= {φ,{b},{c},{d},{a,b},{c,d},{b,c},{b,d},{a,b,c},{a,b,d},{b,c,d},X}.
Therefore, {c},{d},{b,c},{b,d},{a,b,c},{a,b,d} are sg-closed sets but not s-closed.
Also, $T_\alpha = T_s$, $T_\alpha^C = T_s^C$ & $T_{\alpha g}^C = T_{sg}^C$ which show that {c},{d},{b,c},{b,d},{a,b,c},{a,b,d} are αg-closed sets but not α-closed sets.
Similarly, the other cases can be dealt with.

Theorem (4.1)*:* *A space (X,T) is $\wp$ −symmetric if and only if {x} is $\wp$g-closed for each x∈X, where $\wp$ = p,s, α & β.*

Proof: Necessity: Let (X,T) be $\wp$-symmetric , then for distinct points x,y of X ,
y∈ $\wp$ −cl ({x})⇒ x∈ $\wp$ −cl ({y}) where $\wp$ = p,s, α & β.
Let {x}⊂D where D is a $\wp$ −open set in (X,T). Let $\wp$ −cl ({x})⊄ D. This means that ($\wp$-cl ({x}))∩D^c ≠φ. Let y ∈ ($\wp$ −cl ({x}))∩D^c. Now, we have x∈ $\wp$ −cl ({y})⊂D^c and x ∉ D. but this is a contradiction . Hence, $\wp$ −cl ({x})⊂D whenever {x} ⊂D & D is $\wp$ -open. Consequently, {x} is a $\wp$g-closed set.

Sufficiency: Let in a space (X,T) , each {x} is a $\wp$g-closed set where x∈ X. Let x,y∈ X & x ≠ y such that x∈ $\wp$ −cl ({y}) but y∉ $\wp$ −cl ({x}…………(1)
This implies that y∈ ($\wp - cl$ ({x}))c
⇒ {y} ⊂ ($\wp - cl$ ({x}))c
⇒ $\wp - cl$ ({y}) ⊂ ($\wp - cl$ ({x}))^c…………………(2)
as {y} is a $\wp$g-closed set by the assumption.
Now, {x} ⊂ ($\wp$ −cl ({x}))^c from (1) & (2). this is an assumption contradiction which arises due to the acceptance of (1) and consequently , we have
x∈ $\wp$ −cl ({y})⇒ y∈ $\wp$ −cl ({x}) ; for every x ≠y . Therefore, the space (X,T) is $\wp$ −symmetric.
Hence the theorem.
***Corollary* (4.1)**: *If a space (X,T) is $\wp$ −T_1 space , then it is $\wp$ −symmetric, where $\wp$ = p,s, α & β.*

Proof: In a $\wp$-T_1 space, singleton sets *are* $\wp$ −closed by Theorem (1.2) , and therefore $\wp$g-closed by Lemma (4.1). By Theorem (4.1), the space (X,T) is $\wp$ −symmetric, where $\wp$ = p,s, α & β.

Remark (4.1):
The converse of the corollary (4.1) is not necessarily true as shown in the following example:
Let X = {a,b,c,d,e}. T = {φ,{a},{b},{a,b},{c,d},{a,c,d},{b,c,d},(a,b,c,d},{b,c,d,e},X}.
Then
T_s^C ={φ,{a},{b},{e},{a,b},{a,e},{b,e},{c,d},{a,b,e},{a,c,d},{c,d,e},{b,c,d,e}, {a,c,d,e}, X}

T_s ={φ,{b},{a},{a,b},{b,e},{c,d},{a,b,e},{a,c,d},{b,c,d},{c,d,e},{a,b,c,d},{a,c,d,e},
{b,c,d,e},X}.
The space (X,T) is not s-T_1 but s-symmetric.

Theorem (4.2): *for a topological space (X,T) , the following properties are equivalent :*
(a) (X,T) is a$\wp$ −symmetric & $\wp$ −T_0 space;
(b) (X,T) is $\wp$ −T_1 space.

Proof: (a) ⇒(b):
Let x,y ∈ X and x ≠y. Since, (X,T) is a $\wp-T_0$ space, hence we may assume that x ∈ $G_1 \subset \{y\}^c$ for some $\wp-$ open set G_1 . Then x ∉ $\wp-$cl ({y}). Consequently y ∉ $\wp-$cl ({x}). There exists a $\wp-$open set G_2 such that y ∈ $G_2 \subset \{x\}^c$. Therefore, (X,T) is a $\wp-T_1$ space.
(b) ⇒(a):
Corollary (4.1) depicts that (X,T), being $\wp-T_1$ space is $\wp-$symmetric .
Remark (1.1) provides that (X,T) being $\wp-T_1$ space is necessarily $\wp-T_0$ space.
The above two facts together establish that (b) ⇒(a).
Hence ,the theorem.

Corollary (4.2): *If (X,T) is $\wp-$symmetric, then (X,T) is $\wp-T_0$ ⇔ (X,T) is $\wp-T_1$.*

Proof: Here, '⇒" follows from Theorem (4.2) & '⇐' follows from Remark (1.1).

II. Conclusion

An overview of separation axioms by nearly open sets focuses its attention on the literature of $\wp-T_k$(k= 0,1,2 & $\wp$ = p,s, α & β) spaces in the compact form in this paper.

The study of $\wp-R_0$ & $\wp-R_1$ spaces has been enunciated and the related properties are kept ready at a glance. The $\wp$- symmetry of a topological space along with example and basic results has been exhibited at one place.

The future scope of the overview is to compile the literature & research concern with $\wp-T_{1/2}$ spaces($\wp$ = p,s, α & β) and the related fundamental properties & results are to be prepare as a ready reckoning at a glance.

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